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RESEARCH ARTICLE

On asymmetric equilibria in rent-seeking contests with strictly increasing returns

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Abstract
This paper revisits the n-player rent-seeking contest with homogeneous valuations and increasing returns. Our main result says that, for any $m \in \{2, \dots, n-1\}$, there are threshold values $1 < R_*(m) < R^*(m) \leq 2$ for the Tullock parameter R such that a pure strategy equilibrium with m active players exists if and only if $R \in [R_*(m),\ R^*(m)]$. Among other things, this observation leads to a simple characterization of the values of R for which the n-player contest has a unique pure strategy equilibrium.

Keywords Rent-seeking contests · Increasing returns · Asymmetric equilibria · Monotone comparative statics

JEL Classfication C72-Noncooperative games · D72-Political processes: rent-seeking, lobbying, elections, legislatures, and voting behavior · D74-Conflict · Conflict resolution · Alliances · Revolutions

1 Introduction

In the n-player rent-seeking contest (Tullock 1980), a given set of $n \geq 2$ players compete for a rent of value $V > 0$. Let $x_i \geq 0$ denote the effort exerted by contestant $i \in \{1, \dots, n\}$. Normalizing the value of the rent to unity, contestant i's payoff is given as

$$\Pi_i(x_1, \dots, x_n) = \frac{x_i^R}{x_1^R + \dots + x_n^R} - x_i,$$

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where $R > 0$ is the usual parameter, and the ratio is read as $1/n$ if $x_1 = \ldots = x_n = 0$. In any *symmetric* equilibrium,[1]

$$x_1^* = \cdots = x_n^* = \frac{n-1}{n^2} R.$$

This type of equilibrium turns out to exist if and only if active contestants break even, i.e., if and only if

$$R \leq R^*(n) \equiv \frac{n}{n-1}.$$

In any *asymmetric* equilibrium, however, a strict subset consisting of $m \in \{2, \ldots, n-1\}$ active players exert the same positive effort, while the remaining, inactive players exert zero effort (Pérez-Castrillo and Verdier 1992). This type of equilibrium may arise only if the contest technology exhibits increasing returns, i.e., only if $R > 1$. The prediction is that there is a "club" of active rent-seekers, while outsiders are discouraged to participate. An asymmetric equilibrium consequently exists under two conditions, viz. that active players break even, and inactive players find it optimal to stay out. While the first condition is analogous to the parameter restriction for the symmetric equilibrium, the second condition is more intricate and captured by the inequality

$$\frac{R^R\,(m-1)^R}{m^{2R-1}} \geq \frac{(R-1)^{R-1}}{R^R}. \tag{1}$$

Cornes and Hartley (2005) pointed out that, in the relevant domain, condition (1) becomes less stringent as m increases, which intuitively means that keeping outsiders away is easier for larger clubs. This observation leads to useful constraints on m under which an asymmetric equilibrium exists. However, it has to our knowledge not been formally studied how condition (1) depends on the contest technology. Thus, the set of parameter values R for which an asymmetric equilibrium with $m \in \{2, \ldots, n-1\}$ active players exists has not really been well-understood so far.[2]

In this paper, we revisit the n-player Tullock contest with homogeneous valuations and strictly increasing returns. It is shown that, for any $m \in \{2, \ldots, n-1\}$, there exists a lower threshold value $R_*(m) \in (1, R^*(m))$ such that an asymmetric equilibrium with precisely $m < n$ active players exists if and only if $R \in [R_*(m), R^*(m)]$. Our main contribution is, consequently, the formal proof that inequality (1) becomes less demanding as R increases. Intuitively, with a larger R, competition for the rent within the club becomes tighter, making it harder for outsiders to enter. Our main result therefore clarifies the nature of the conditions for the existence of an asymmetric pure strategy equilibrium in the n-player contest.

The analysis is extended in three ways. First, we show that the lower bound $R_*(m)$ is strictly decreasing in m. Given that the same is obviously true for the upper bound

[1] In this paper, we focus on equilibria in pure strategies.

[2] For example, Ryvkin (2007, Sec. 3) offered valuable intuition and numerical illustration, though without formal proofs.

$R^*(m)$, this means that the closed interval $[R_*(m), R^*(m)]$ in the parameter space over which asymmetric equilibria with precisely $m < n$ active players exist is shifting downwards as m goes up. Second, we show that

$$R^*(m+1) > R_*(m) \tag{2}$$

for all $m \geq 2$. This fact implies that the respective intervals in the parameter space over which an asymmetric equilibrium with $m \in \{2, \ldots, n-1\}$ active players exists jointly cover the interval $[R_*(n-1), 2]$. Given that the symmetric equilibrium exists for $R \leq R^*(n)$ with $R^*(n) > R_*(n-1)$, one arrives at an alternative proof of an important existence result for pure strategy equilibria in rent-seeking contests with $R \leq 2$ (Cornes and Hartley 2005, Cor. 2). Third and finally, we show that the condition $R < R_*(n-1)$ is both necessary and sufficient for the existence of a unique pure strategy equilibrium in the n-player contest. In sum, these results provide a comprehensive characterization of the equilibrium set of the Tullock contest with homogeneous valuations and increasing returns.

The remainder of this paper is structured as follows. Section 2 reviews prior work. Section 3 states our main result. Section 4 offers extensions. The Appendix contains a technical proof.

2 Review of prior work

The existing literature has characterized the best-response correspondence as well as the conditions for the existence of asymmetric pure strategy equilibria with $m < n$ active players, as summarized in the following proposition.

Proposition 1 (Pérez-Castrillo and Verdier 1992; Cornes and Hartley 2005) *Suppose that $R > 1$. Then, the following holds:*

(i) *Being active is a best response for contestant i if and only if*[3]

$$\sum_{j \neq i} x_j^R \in \left(0, \frac{(R-1)^{R-1}}{R^R}\right].$$

(ii) *In any equilibrium with precisely $m \geq 2$ active players, $x_i^* = \frac{m-1}{m^2} R$, for any active contestant i.*

(iii) *An equilibrium with $m \in \{2, \ldots, n-1\}$ active contestants exists if and only if $R \leq 2$ and*

$$m \in \{m_*(R), \ldots, m^*(R)\},$$

where $m_*(R)$ is the lowest integer satisfying inequality (1), and $m^*(R)$ is the largest integer satisfying $m \leq \frac{R}{R-1}$; moreover, $m_*(R) \leq m^*(R)$.

[3] For $\sum_{j \neq i} x_j^R = 0$, there is no best response.

Table 1 Bounds for the parameter m

R	$m_*(R)$	$m^*(R)$
1.05	8	20
1.10	5	10
1.15	4	7
1.20	3	6
1.25	3	5
1.30	3	4
1.35	3	3
1.40	2	3
1.45	2	3
1.50	2	3
1.55	2	2
1.60	2	2

Proof (i) See Pérez-Castrillo and Verdier (1992, Prop. 1). (ii) See Pérez-Castrillo and Verdier (1992, Prop. 3). (iii) See Cornes and Hartley (2005, Thm. 7 and Lem. 1). □

Table 1 shows $m_*(R)$ and $m^*(R)$ for selected values of R. For example, in the n-player contest with $R = 1.25$, there exists an asymmetric equilibrium with precisely $m < n$ active players if and only if $m \in \{m_*(R), \ldots, m^*(R)\} = \{3, 4, 5\}$. Thus, in the example, there are no asymmetric equilibria if $n = 3$, while there are asymmetric equilibria with $m \in \{3, 4\}$ active players if $n = 5$.[4]

Proposition 1 allows understanding how relationship (1) captures the equilibrium condition for inactive players. The right-hand side of the inequality corresponds to the activity cutoff specified in part (i) of the proposition, while the left-hand side of the inequality corresponds to the aggregate $\sum_{i=1}^{m} x_i^R$ that results from the equilibrium efforts characterized in part (ii). Inactivity is optimal if and only if the left-hand side weakly exceeds the right-hand side.

The proposition above is useful in these and other ways. However, as has been explained in the Introduction, the characterization of the equilibrium set accomplished in prior work remains partial because it does not allow to easily characterize the range of R for which an asymmetric equilibrium with m active players exists in the n-player rent-seeking contest.[5]

3 Main result

The main result of the present paper is the following.

[4] As mentioned before, the symmetric equilibrium exists if and only if $R \leq \frac{n}{n-1}$, which is equivalent to $n \leq \frac{R}{R-1}$ and, hence, to $n \leq m^*(R)$. Table 1 therefore provides information also about symmetric equilibria. For $R = 1.25$, for instance, the symmetric equilibrium exists if and only if $n \in \{2, \ldots, m^*(R)\} = \{2, 3, 4, 5\}$.

[5] Similarly, the monotonicity of $m_*(R)$, suggested by Table 1, has not been studied in prior work. See Sect. 4.2.

Proposition 2 *There exists a lower threshold value* $R_*(m) \in (1, R^*(m))$ *such that an asymmetric equilibrium with precisely* $m \in \{2, \ldots, n-1\}$ *active contestants exists if and only if* $R \in [R_*(m), R^*(m)]$.

Proof As has been discussed in the Introduction, an asymmetric equilibrium with precisely $m \in \{2, \ldots, n-1\}$ active contestants exists in the n-player rent-seeking contest if and only if (i) active players have no incentive to deviate, and (ii) inactive players find it optimal to stay out. The first condition boils down to the break-even requirement $R \leq R^*(m)$.[6] As for the second condition, suppose without loss of generality that contestants $i \in \{1, \ldots, m\}$ are active. Then, by Proposition 1,

$$x_1^* = \ldots = x_m^* = \frac{m-1}{m^2} R.$$

Therefore,

$$\sum_{i=1}^{m} x_i^R = \frac{(m-1)^R}{m^{2R-1}} R^R,$$

so that remaining inactive is optimal for any contestant $i \in \{m+1, \ldots, n\}$ if and only if inequality (1) holds. Taking the logarithm, the inequality is seen to be equivalent to

$$\phi(R) \equiv 2R \ln R + R \ln(m-1) - (R-1)\ln(R-1) - (2R-1)\ln m \geq 0.$$

In the limit $R \to 1$, we have $(R-1)\ln(R-1) \to 0$, so that $\lim_{R \to 1} \phi(R) < 0$. On the other hand, at $R = R^*(m)$, any active contestant has a payoff of zero. Clearly, then, an inactive contestant can only enter with losses. Thus, $\phi(R^*(m)) > 0$. Note further that

$$\begin{aligned}\frac{\partial \phi(R)}{\partial R} &= 2\ln R - 2\ln m - \ln(R-1) + \ln(m-1) + 1 \\ &= \ln\left(\frac{R^2(m-1)e}{(R-1)m^2}\right),\end{aligned}$$

where $e = \exp(1) \approx 2.71828$. To establish monotonicity of ϕ, it therefore suffices to show that

$$\frac{R^2(m-1)e}{(R-1)m^2} > 1, \tag{3}$$

[6] For a direct proof, note that

$$\frac{\partial^2 \Pi_i(x_1, \ldots, x_n)}{\partial x_i^2} = \frac{R x_i^{R-2} B}{(B + x_i^R)^3}\Big((R-1)B - (R+1)x_i^R\Big),$$

provided that $B = \sum_{j \neq i} x_j^R > 0$. Thus, if at least one opponent is active, there exists $x_i^\# > 0$ such that player i's objective function is strictly convex for $x_i \leq x_i^\#$ and strictly concave for $x_i \geq x_i^\#$. For such an objective, however, the first-order condition together with the break-even requirement is equivalent to global optimality.

Table 2 Bounds on the parameter R

m	R_*(m)	R^*(m)
2	1.35050	2.00000
3	1.16531	1.50000
4	1.10848	1.33333
5	1.08079	1.25000
6	1.06438	1.20000
7	1.05352	1.16667
8	1.04580	1.14286
9	1.04002	1.12500
10	1.03555	1.11111

for any $R \in (1, R^*(m)]$. We know that $\frac{R}{R-1} \geq m$. Hence, it is sufficient to show that $\frac{R(m-1)}{m} > \frac{1}{e}$, which is clearly the case. Therefore, there indeed exists a threshold value $R_*(m) \in (1, R^*(m))$ with the stated property. □

Table 2 shows the values of $R_*(m)$ and $R^*(m)$ for $m \in \{2, \ldots, 10\}$. For instance, an asymmetric equilibrium with $m = 3$ active players exists in a contest with $n > 3$ players if and only if $R \in [R_*(3), R^*(3)] = [1.16531, 1.50000]$.[7]

4 Extensions

In this section, we discuss the comparative statics (Sect. 4.1), an alternative proof of an existence result in Cornes and Hartley (2005) (Sect. 4.2), and conditions for the uniqueness of the equilibrium (Sect. 4.3).

4.1 Comparative statics

As noted before, the upper bound $R^*(m) = \frac{m}{m-1}$ is strictly decreasing in m, starting from $R^*(2) = 2$ and approaching 1 as $m \to \infty$. The following result shows that the comparative statics of the lower bound is similar.

Proposition 3 *$R_*(m)$ is strictly decreasing in m, with* $\lim_{m\to\infty} R_*(m) = 1$.

Proof To see why $R_*(m)$ is strictly monotone in m, recall from the proof of Proposition 2 that $R = R_*(m)$ solves the equation $\phi(R) = 0$. Considering m for the moment as a continuous variable, implicit differentiation shows that

$$\frac{dR_*(m)}{dm} = -\frac{\partial\phi/\partial m}{\partial\phi/\partial R} = -\frac{1-(m-2)(R-1)}{m\,(m-1)\ln\frac{R^2(m-1)e}{(R-1)m^2}}.$$

[7] Notably, the symmetric equilibrium exists for any $R \leq R^*(n)$, i.e., even if $R < R_*(n)$. The reason for the relaxed conditions in the case $n = m$ is that there are no potential entrants around, which makes it easier to have the equilibrium.

Now, from $R = R_*(m) < R^*(m) = \frac{m}{m-1}$, it follows that $1 - (m-2)(R-1) > 0$. By (3), also the denominator is positive in the relevant domain. Therefore, $dR_*(m)/dm < 0$, as has been claimed. The limit property for $R_*(m)$ follows from $R_*(m) \in (1, R^*(m))$ and $\lim_{m\to\infty} R^*(m) = 1$. □

As has been discussed in the Introduction, Proposition 3 implies that the boundaries of the interval $[R_*(m), R^*(m)]$ are strictly decreasing in m. Numerically, this is evident from Table 2.

Essentially the same argument shows that $m_*(R)$ is decreasing in R. Thus, the bounds of the range $\{m_*(R), \ldots, m^*(R)\}$ are monotonically decreasing in R, which complements Proposition 1.

4.2 An alternative proof of Cornes and Hartley (2005, Cor. 2)

Cornes and Hartley (2005) observed that, regardless of the number of players $n \geq 2$, a pure strategy equilibrium exists if and only if $R \leq 2$. The original proof is constructive. An alternative proof is presented below.

Proposition 4 (Cornes and Hartley 2005) *An equilibrium exists in the n-player rent-seeking contest if and only if $R \leq 2$.*

Proof The argument is standard for $R \leq 1$. Take some $R > 1$. Then, as noted in the Introduction, a symmetric equilibrium exists for any $R \leq R^*(n)$. Since $R^*(2) = 2$, this proves the claim for $n = 2$. Suppose that $n \geq 3$. In the Appendix, we show that $R_*(m) < R^*(m+1)$, for any $m \geq 2$. Evaluating at $m = n-1$, an equilibrium exists for $R \leq R_*(n-1)$. By Proposition 2, an asymmetric equilibrium with $(n-1)$ active contestants exists for $R \in [R_*(n-1), R^*(n-1)]$. Hence, there is some equilibrium for any $R \leq R^*(n-1)$. This proves the claim for $n = 3$ and, by straightforward induction, for any $n \geq 2$. □

4.3 Equilibrium uniqueness

Cornes and Hartley (2005) derived the conditions under which the equilibrium in the n-player rent-seeking contest is unique. The following result restates those conditions somewhat more explicitly as a constraint on the parameter R.

Proposition 5 *In the n-player Tullock contest with homogeneous valuations, the symmetric equilibrium is the unique pure strategy equilibrium if and only if $R \in (0, R_*(n-1))$.*

Proof The uniqueness of the pure strategy equilibrium for $R \leq 1$ is again standard. For $R > 1$, the symmetric equilibrium is unique if and only if, for any $m \in \{2, \ldots, n-1\}$, there is no asymmetric equilibrium, or using Proposition 2, if and only if $R \notin [R_*(m), R^*(m)]$. By Proposition 3, $R_*(n-1) < \ldots < R_*(2)$. Hence, for $R < R_*(n-1) < R^*(n)$, the symmetric equilibrium is indeed the unique equilibrium. Next, from, $R^*(2) = 2$ and the proof of Proposition 4, it follows that, for any $R \in [R_*(n-1), 2]$, there do exist asymmetric equilibria. This proves the claim. □

For illustration, consider again the simplest case where $n = 3$. The symmetric equilibrium exists for $R \le R^*(3) = 1.50000$. By Proposition 2, however, an asymmetric equilibrium with two active players exists if and only if $R \in [R_*(2), R^*(2)] = [1.35050, 2.00000]$. Thus, there is a unique equilibrium in the contest with $n = 3$ players if and only if $R < R_*(2) = 1.35050$.

A Appendix

This appendix contains material omitted from the proof of Proposition 4. Specifically, we will verify inequality (2) for any given $m \ge 2$. Given the strict monotonicity property of the function $\phi(R)$ established in the proof of Proposition 2, it is sufficient to check that inequality (1) holds strictly at $R = R^*(m+1) = \frac{m+1}{m}$. Substituting gives

$$\frac{R^R(m-1)^R}{m^{2R-1}} = \frac{\left(\frac{m+1}{m}\right)^{\frac{m+1}{m}}(m-1)^{\frac{m+1}{m}}}{m^{\frac{m+2}{m}}} = \frac{(m^2-1)^{\frac{m+1}{m}}}{m^{\frac{2m+3}{m}}},$$

and

$$\frac{(R-1)^{R-1}}{R^R} = \frac{m^{-\frac{1}{m}}}{\left(\frac{m+1}{m}\right)^{\frac{m+1}{m}}} = \frac{m}{(m+1)^{\frac{m+1}{m}}}.$$

Hence, the strict version of inequality (1) is reduced to

$$\left((m^2-1)(m+1)\right)^{\frac{m+1}{m}} > m^{\frac{3m+3}{m}}.$$

Raising both sides to the power $\frac{m}{m+1}$ simplifies it to $(m^2-1)(m+1) > m^3$, which factors as $m(m-1) > 1$. Since this holds for any $m \ge 2$, the proof is complete.

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